Assignment 3

1. (Revised) Establish the identity

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \frac{\cos 8x}{48} + \cdots\right), \quad x \in [0, \pi].$$

   How to explain an odd function is now expressed as the sum of even functions?

2. Show that there is a countable subset of $C[a,b]$ such that for each $f \in C[a,b]$, there is some $\varepsilon > 0$ such that $\|f - g\|_\infty < \varepsilon$ for some g in this set. Suggestion: Take this set to be the collection of all polynomials whose coefficients are rational numbers.

3. Let f be continuously on $[a,b] \times [c,d]$. Show that for each $\varepsilon > 0$, there exists a polynomial $p = p(x,y)$ so that

$$\left\|f - p\right\|_\infty < \varepsilon, \quad \text{in } [a,b] \times [c,d].$$

   In fact, this result holds in arbitrary dimension.

4. Let $\{\varphi_k\}, k \geq 1$, be an orthonormal set $R[a,b]$. Show that for every $f \in R[a,b]$,

$$\sum_k < f, \varphi_k >_2^2 \leq \int_a^b f^2.$$

   This is called Bessel inequality.

5. Same as in the previous problem but now the index set $\mathcal{A}$ could be arbitrary, for instance, an uncountable set. Show that for each $f \in R[a,b]$, there exists a countable subset $\mathcal{B}$ from the index set such that $< f, \varphi_\alpha >_2 = 0$, for all $\alpha \in \mathcal{A} \setminus \mathcal{B}$. Hint: Show that the set $\{\beta \in \mathcal{A} : < f, \varphi_\alpha >_2 \geq 1/k\}$ is a finite set for each $k \geq 1$.

6. (a) Let S be the vector subspace in $C[0,1]$ spanned by the polynomials $1, x$ and x^2. Find an orthonormal set in S which spans S.

   (b) Find the quadratic polynomial that minimizing the L^2-distance from $1/(1+x)$ to S.

7. The Legendre polynomials are given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n\left[\left(x^2 - 1\right)^n\right]}{dx^n}, \quad n \geq 0 \ .$$

   (a) Write down $P_0, \cdots P_4$.

   (b) Show that

$$\left\{\sqrt{\frac{2n+1}{2}}\, P_n\right\}_{n=0}^{\infty}$$

   forms an orthonormal set in $R[-1,1]$.

   (c) Verify that each P_n is a solution to the differential equation

$$\left(\left[(1 - x^2)\right] y'\right)' + n(n+1)y = 0.$$

8. Let $f, g \in R_{2\pi}$. Show that

$$\int_{-\pi}^{\pi} fg = 2\pi a_0 c_0 + \pi \sum_{n=1}^{\infty} (a_n c_n + b_n d_n),$$

   where a_n, b_n and c_n, d_n are respectively the Fourier coefficients of f and g.

9. Establish the following identities:

   (a)
   $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96},$$

   (b)
   $$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960},$$

   (c)
   $$\sum_{n=0}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

   Hint: Use the function $f(x) = |x|$ and the odd function $g(x) = x(\pi - x)$.